A beamline zero-degree spectrometer for measurements of projectile fragment distributions

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A technique for using a section of a beamline as a zero-degree spectrometer is described. The beamline magnets of the K1200 cyclotron at NSCL were operated as a zero-degree medium acceptance spectrometer to measure projectile fragment distributions. Projectile fragments were produced in a fragmentation target at the exit of the cyclotron and detected at the end of the beamline with a detector array consisting of two position-sensitive detectors and a \( \Delta E - E \) telescope. Energy loss, position and angle at the focal plane and time of flight were measured for the isotopes produced. A procedure was developed to calculate the reaction angle and the momentum of the isotopes from the measured quantities. Typical examples of momentum and angular distributions of projectile fragments from a \( ^{14}\text{N} \) beam at 75 MeV/u with Al and Ta targets at and near 0° are shown.

I. INTRODUCTION

Projectile fragmentation is a very important mechanism for the production of nuclei far from stability. The exotic nuclei produced in this way can be separated and used as secondary (radioactive) beams, allowing the possibility of a new class of experiments to study nuclear structure and nuclear reaction mechanisms. The technique of converting a primary beam into a secondary radioactive beam by projectile fragmentation has been successfully used at LBL,\(^1\) GANIL,\(^2\) and RIKEN\(^3\) and similar devices are in the final construction stages for other labs (e.g., GSI\(^4\)). At the National Superconducting Cyclotron Laboratory (NSCL), this technique is going to be realized by using a beam analysis device, the A1200.\(^5\)

Projectile fragments are produced in peripheral collisions at intermediate and high energies. At high energies, above 200 MeV/u, there is a rather complete and systematic data set of the fragmentation distributions that have been successfully described by several models. In general, these models are based on the idea of a two-step process consisting of a fast collision step and a slow de-excitation one. At intermediate energies, from 50 to 200 MeV/u, there are only a few scattered measurements and no parametrization of the cross sections or understanding of the reaction mechanisms. In order to understand the reaction mechanisms of peripheral collisions at intermediate energies and obtain parametrizations of their distributions, refined measurements around 0° are necessary (since the projectile-like isotopes are produced in a narrow cone about 0°). Such systematic data and parametrizations, apart from a nuclear reaction standpoint, will improve our predictive power to choose the proper energy and beam-target combinations to produce a desired radioactive beam.

Measurements of reaction products at 0° are difficult in general, because of the presence of the primary beam and require the use of magnetic spectrometers. In this article we discuss a technique where, instead of a usual spectrometer, a section of a beamline containing one bending element was used for 0° measurements. We describe an experiment that we performed at NSCL in which the interim beamline\(^6\) from the K1200 cyclotron to a 92-inch diameter cylindrical scattering chamber\(^7\) was operated as a 0° spectrometer. Specifically, in this article, after a description of the beamline, its characteristics, and the detector system used, we discuss the measurements we performed with this system, the procedures we developed for the analysis of the data, and we show several representative results of projectile fragment distributions obtained from these data. Finally, in the Appendix, we include a derivation of the equations developed for the present study. Such techniques could be implemented at any laboratory with modest bending magnets in the beam transport system.

II. DESCRIPTION OF THE SPECTROMETER AND THE DETECTORS

The interim K1200 beamline consisted of four superconducting magnets (Fig. 1), which are a quadrupole singlet, a quadrupole doublet, a \( \pm 16^\circ \) bending dipole magnet, and a quadrupole doublet. The details of these magnets have been described previously.\(^8\) For the present study, we operated the two quadrupole doublets and the bending magnet as a 0° spectrometer (the quadrupole singlet was turned off). The characteristics of the magnetic elements of the spectrometer are summarized in Table I.

Projectile fragments, produced in a fragmentation target placed 40 cm after the exit of the K1200 cyclotron, traveled through the device and were focused and detected inside the 92-in. scattering chamber. The detection system for the present work was mounted on the movable radial arm of the chamber and consisted of two position counters.
and a $\Delta E - E$ plastic scintillator telescope. The first position counter was placed near the entrance to the 92-in. scattering chamber and the second counter 1.8 m further downstream. Both position counters were low pressure multiwire proportional counters (LP-MWPCs) of Breskin type.\textsuperscript{3} From these detectors, a position signal is obtained by a delay-line readout of the electric charge induced on two layers of interleaved cathode strips. The time difference between the signals from the two ends of the delay line is measured and is proportional to position along the counter. The position measurement from the front counter contained information on the rigidity, whereas the difference between the position measurements of the two counters contained information on the scattering angle of a particle. Section V describes in detail how the rigidity and the scattering angle were obtained from the measured position and angle. The $\Delta E - E$ telescope, mounted behind the back position detector, consisted of a thin (3 mm) fast plastic scintillator for energy loss measurement and a thick (5 cm) plastic scintillator as a stopping detector. Each scintillator was viewed by a photomultiplier tube (PMT) on each end (left or right). A $\Delta E$ parameter was calculated event-by-event by taking the square root of the product of the left and right PMT signals of the $\Delta E$ scintillator. Due to light attenuation by self-absorption in the plastic scintillator, each PMT signal is exponentially dependent on the position of a particle along the scintillator. The product of the two signals is position independent and its square root proportional to $\Delta E$ (whereas the logarithm of the ratio of these signals is linearly related to position). The atomic number $Z$ of the particles was obtained from the $\Delta E$ parameter, since $\Delta E \propto Z^2/v^2$, where $v$ is the velocity of the particles.

The back LP-MWPC had a low efficiency, so its position information was not used in the analysis of the data. Instead, a back position parameter was constructed from the logarithmic ratio of the two PMT signals of the $\Delta E$ scintillator. This position measurement was found to be consistent with the position measurement provided by the back LP-MWPC for highly ionizing particles and was combined with the front position parameter to provide a final angle parameter for the particles. The position resolution of both LP-MWPCs was 1 mm; the position resolution of the back position parameter obtained by the $\Delta E$ scintillator was 1.0 cm and the angle resolution obtained by the combination of this position parameter and the front position parameter was 6 mr. The fact that the angle measurement had to rely on the less accurate position values

TABLE I. Characteristics of the spectrometer and its elements.

<table>
<thead>
<tr>
<th>Quadrupoles</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective length</td>
<td>43.0 cm</td>
</tr>
<tr>
<td>Pole-tip radius</td>
<td>5.08 cm</td>
</tr>
<tr>
<td>Max. magnetic field at pole tip</td>
<td>15 kG</td>
</tr>
<tr>
<td>Electric current at max. field</td>
<td>14 A</td>
</tr>
<tr>
<td>Distance between quads in a doublet</td>
<td>16.7 cm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dipole</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean deflection angle</td>
<td>16.4°</td>
</tr>
<tr>
<td>Mean bending radius</td>
<td>3.08 m</td>
</tr>
<tr>
<td>Entrance face angle</td>
<td>0°</td>
</tr>
<tr>
<td>Exit face angle</td>
<td>16.4°</td>
</tr>
<tr>
<td>Effective length of central trajectory</td>
<td>0.882 m</td>
</tr>
<tr>
<td>Pole-gap width</td>
<td>5.0 cm</td>
</tr>
<tr>
<td>Maximum magnetic field</td>
<td>18 kG</td>
</tr>
<tr>
<td>Electric current at max. field</td>
<td>100 A</td>
</tr>
<tr>
<td>Distances between elements</td>
<td></td>
</tr>
<tr>
<td>Target—first doublet entrance</td>
<td>1.187 m</td>
</tr>
<tr>
<td>First doublet exit—dipole entrance</td>
<td>1.677 m</td>
</tr>
<tr>
<td>Dipole exit—second doublet entrance</td>
<td>0.852 m</td>
</tr>
<tr>
<td>Second doublet exit—front LP-MWPC</td>
<td>2.662 m</td>
</tr>
<tr>
<td>Front LP-MWPC—back LP-MWPC</td>
<td>1.840 m</td>
</tr>
</tbody>
</table>
TABLE II. Important parameters of the spectrometer.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal magnification $(x</td>
<td>x) = -0.63$</td>
</tr>
<tr>
<td>Dispersion at the front LP-MWPC $(x</td>
<td>\delta) = 0.65 \text{ cm} / %$</td>
</tr>
<tr>
<td>Horiz. beam spot at the target (full width) $\Delta x_0 = 6 \text{ mm}$</td>
<td></td>
</tr>
<tr>
<td>Max. horiz. angular acceptance $\Delta \delta_0 = 35 \text{ mrad}$</td>
<td></td>
</tr>
<tr>
<td>Max. vertical angular acceptance $\Delta \delta_0 = 10 \text{ mrad}$</td>
<td></td>
</tr>
<tr>
<td>Max. solid angle $\Delta \Omega = 0.35 \text{ msr}$</td>
<td></td>
</tr>
</tbody>
</table>

**First order TRANSPORT matrix elements**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>l</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.63119</td>
<td>-0.07302</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>18.66188</td>
<td>0.57452</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>1.06086</td>
<td>0.89927</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>1.53741</td>
<td>1.39669</td>
<td>0.00000</td>
</tr>
<tr>
<td>1.24998</td>
<td>0.04176</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>1.00000</td>
<td>-0.01198</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

**Second- and third-order matrix elements**

| $(x|x)\delta$ = $-1.18 \times 10^{-1}$ | $(\theta|\delta) = -6.65 \times 10^{-1}$ |
| $(x|\delta) = -4.53 \times 10^{-3}$ | $(\theta|\delta) = -5.07 \times 10^{-2}$ |
| $(x|\theta) = -5.8 \times 10^{-5}$ | $(\theta|\theta) = -2.3 \times 10^{-4}$ |
| $(x|\delta^2) = 1.02 \times 10^{-5}$ | $(\theta|\delta^2) = 1.58 \times 10^{-2}$ |
| $(x|\delta^3) = 2.78 \times 10^{-4}$ | $(\theta|\delta^3) = 1.28 \times 10^{-3}$ |

Magnetic fields (in kG) for $^{14}$N at 75 MeV/u

<table>
<thead>
<tr>
<th>First quad</th>
<th>6.204</th>
<th>Third quad</th>
<th>1.701</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second quad</td>
<td>-3.667</td>
<td>Fourth quad</td>
<td>-0.041</td>
</tr>
<tr>
<td>Dipole</td>
<td>8.244</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

from the plastic scintillator had a significant effect on the overall momentum resolution (see Sec. III).

Time of flight (TOF) was measured relative to the K1200 rf cycle. A start timing signal was obtained from the $\Delta E$ scintillator. A combination of TOF and position provided an $A/q$ parameter, which was combined with the $\Delta E$ parameter in a two-dimensional histogram to provide particle identification. Due to multiple turn extraction from and the phase acceptance of the cyclotron, only limited TOF resolution (approximately 4 ns) was obtained, which hindered the particle identification for $Z_s$ higher than 6.

**III. BEAM OPTICS CALCULATIONS**

Beam optics calculations (up to third order), performed with the program TRANSPORT, showed that a moderate acceptance solution was possible for this simple spectrometer, i.e., 35-mr horizontal and 10-mr vertical angular acceptances, and 8% momentum acceptance. The optical characteristics of the spectrometer, the first-order TRANSPORT matrix elements, and the important second- and third-order matrix elements at the front position counter are summarized in Table II. The TRANSPORT beam optics notation and units have been used in Table II and throughout this article. At the end of Table II, the field values of the magnets corresponding to setting the spectrometer for $^{14}$N$^7^+$ at 75 MeV/u (which has a momentum-over-charge ratio $P/q$ of 0.762 GeV/c/e) are given.

For the present system, there are no first-order correlations between the coordinates of the particle motion on the horizontal (bending) $xz$ plane and the vertical (non-bending) $yz$ plane. However, such correlations arise in second and higher orders. The TRANSPORT calculation showed that the second- and third-order correlations of horizontal and vertical coordinates are negligible for the relatively small vertical angular acceptance of 10 mrad in all cases, the contributions to the position and angle are less than 1 mm and 1 mrad, respectively, and these contributions were neglected in the present study. Hence, only the correlations of the coordinates in the bending plane were taken into account when extracting the momentum and the scattering angle of a particle from the measured position and angle. Simple algebraic formulas and a calculation procedure were developed for this purpose and are described in Sec. V.

From the first-order TRANSPORT matrix, we see that a relatively small value of the horizontal magnification $(x|x)$ is achieved, but a considerable $(x|\theta)$ term is present at the front counter, which introduces a first-order correlation of final position with initial scattering angle. Consequently, a first-order horizontal focus was only approximately created at the front detector. The actual focal plane [where $(x|\theta)$ is zero] lies between the two detectors.

In any spectrometer with relatively large momentum and angular acceptance there will be large second-order aberrations [mainly due to the $(x|\theta^2)$ term]. These can be corrected with sextupoles or by focal plane reconstruction. In the present study, we chose the later method. Consequences of this choice are that it is not necessary to have a first-order focus at a position-sensitive detector, but an accurate angular measurement is required.

Generally, it is desirable that the contributions of the second- and higher-order matrix elements at the focal plane should be small fractions (i.e., less than 0.1) of the displacement due to the dispersion. When one position counter is used at the exit of a spectrometer, it is necessary that this detector be placed at the position of the focal plane. However, if two position detectors are used (thus final position and angle of the particles can be measured, as in the present system), it is not necessary to place one of the detectors along the focal plane. In such a case, the position of the focal plane can, in principle, be reconstructed offline, provided that the angle measurement has sufficient accuracy.

For the present system, where reaction angle and momentum reconstruction are performed, an experimental uncertainty in the reaction angle of approximately 15 mrad was obtained from the elastic scattering data. This value is consistent with a conservative estimate of 22 mrad obtained from Eq. (A5), which takes into account the contribution from the final angle uncertainty and the beam-spot size (approximately 6 mm). Similarly, the experimental momentum resolution of approximately 2.0% can be mainly attributed to the reaction angle uncertainty combined with...
the presence of the \((x \theta)\) term. An estimate of the momentum resolution of approximately 2.0%, based on the beam optics, can be obtained from Eqs. (A3) and (A4). However, the momentum resolution of the system with only the front counter position, without scattering angle reconstruction, would be worse. Specifically, at the full horizontal angular acceptance of \(\Delta \theta_b = 35\) mr and momentum acceptance of \(\Delta \beta = 8\%\) of the spectrometer, the effects of the \((x \theta)\) and \((x \theta \delta)\) terms to the image width would result in a momentum resolution of approximately 4.5%.

**IV. EXPERIMENT**

As a test of the beamline spectrometer, we measured the products from the reaction of a \(^{14}\text{N}\) beam at 75 MeV/u with Al and Ta targets with thicknesses of 8 and 10 mg/cm\(^2\), respectively. Data were collected for approximately 10 field settings in steps of 5%, covering the \(A/q\) region of 2.0 to 3.0. The magnetic field of the dipole was measured with an NMR magnetometer. The position and angle calibration of the detectors was performed in two independent ways, giving consistent results: first the radial arm (on which the detecting system was mounted) was rotated by a known angle and the change in position of the \(0^\circ\) elastically scattered beam particles was measured, and second, a slotted copper mask was inserted between the two counters and was irradiated with a dispersed primary beam.

**V. DATA ANALYSIS AND RESULTS**

In the present \(0^\circ\) spectrometer, the momentum and the scattering angle of a particle leaving the target are related to the measured position and angle at the front position counter. The calculations and the procedure that we have developed to extract these quantities from the data are described below.

We can write the magnetic rigidity \(B_p\) of a particle moving in the magnetic field of a dipole as \(^{11}\):

\[
B_p = \frac{P}{q} \cos \theta, \tag{1}
\]

where \(B\) is the magnetic field induction (in kG), \(P\) is the radius (in cm) of the trajectory of the particle in the dipole field, \(q\) is the momentum of the particle (in MeV/c), and \(q\) is its charge (in units of elementary charge). In the present experiment, the position of the undeflected beam particles at several charge states (\(^{14}\text{N}^7^+\) and \(^{14}\text{N}^8^+\) ) obtained at several different magnetic field settings, directly calibrated the curvature radius \(\rho\) versus position (Fig. 2) corresponding to a scattering angle \(\theta_b\) with respect to the spectrometer optical axis at the target position (Fig. 3(a)). This angle, made by the beam emerging from the accelerator, was found to be \(\theta_b = 1^\circ\). We had no control over this angle, since the spectrometer was fixed and there were no bending elements after the cyclotron and before the target. However, it was exploited to increase the observed reaction-angle range to \(2^\circ\).

\[
\rho(\theta_b) = A_0(\theta_b) - A_1(\theta_b)X_{FP}, \tag{2}
\]

where \(X_{FP}\) is the position (in channel number) of a particle at the front position counter. Two additional calibration points, obtained in a subsequent experiment with a \(^{20}\text{Ne}\) beam at 65 MeV/u, agree very well with the nitrogen data. So, for a particle emitted at an angle \(\theta_b\) relative to the spectrometer axis, Eqs. (1) and (2) relate its momentum to the position on the front counter. But for angles \(\theta_b\) different from \(\theta_p\), the correlation of front counter position with angle \(\theta_0\) produced in the spectrometer, renders Eq. (2) inadequate. A \(\theta_p\) versus front counter position calibration would be necessary, which we were not able to obtain in the present experiment. There are at least two ways to perform this calibration; first by bending the beam before the target with an inflector magnet system and second, by inserting a slotted aperture at a certain distance after the target. In the present work, the calibration was obtained from a TRANSPORT calculation, which should be adequate, as the fields of the magnets and their positions along the beamline are well known. In order to take into account the angle dependence of the curvature radius, a relation similar to Eq. (2) was derived in which the coefficients \(A_0\) and \(A_1\) were expressed as functions of the angle \(\theta_0\) in first order. Specifically we can write, similar to Eq. (2):

\[
\rho(\theta_0) = A_0(\theta_0) - A_1(\theta_0)X_{FP}, \tag{3}
\]

where now...
where the coefficients $K_0$ and $K_1$ are related to the calculated first-, second-, and third-order matrix elements with position detector relative to the reference beam and $C_0(\theta_b), C_1(\theta_b), C_2(\theta_b)$ are the coefficients of a quadratic fit of the relation $\theta_{FP} = f(\delta)$ for the 0° elastically scattered beam particles (that is, for $\theta_0 = \theta_b$). So, the scattering angle of a given particle can be calculated according to Eq. (11) from its final angle $\theta_{FP}$ and its momentum deviation $\delta$.

Using these expressions and procedure, the curvature radius (thus the momentum) and the reaction angle of the particles were calculated on an event-by-event basis. The resolutions (FWHM) of the curvature radius and the reaction angle of the beam particles were calculated on an event-by-event basis. The resolutions (FWHM) of the curvature radius and the reaction angle of the 0° elastically scattered beam particles of one run that was chosen as reference can be calculated by the formula:

$$B_p = 3.3556(\gamma A m_p v/q),$$  \hfill (12)
Isotope separation and identification were performed using a two-dimensional histogram of $\Delta E$ vs $A/q$. In this histogram, the groups corresponding to the elastically scattered beam particles were used to calibrate the $A/q$ axis. For each isotope, a $\theta$, vs $P/q$ histogram was generated, $\theta$, cuts were made at $0^\circ$, $1^\circ$, $2^\circ$, (each with width of $1^\circ$) and the corresponding $P/q$ spectra were constructed. Then, for each isotope, after normalization and appropriate combination of the spectra from different runs, a $P/A$ spectrum at each of these angles was obtained. Finally, angular distributions were also constructed by integrating the $P/A$ distributions at each angle.

As representative results of the present study, the momentum distributions of $^{12}$C, $^{15}$C, and $^{10}$Be produced by the $^{14}$N beam at 75 MeV/u on Al target and their angular distributions from Al and Ta targets are shown in Figs. 4 and 5, respectively. The error bars are statistical and are omitted when they are smaller than the plotting symbols. We conservatively estimate that a factor of approximately accounts for the systematic errors primarily due to the uncertainty in the normalization of different runs with overlapping momenta (approximately $\pm 20\%$) and to uncertainties in the beam current integration.

The present results exhibit the main characteristics of momentum and angular distributions of projectile-like isotopes produced at intermediate energies at and near $0^\circ$. Near-projectile fragments (one or two nucleons removed), like $^{13}$C, have narrow momentum distributions peaked near (slightly lower than) the beam velocity. Their angular distributions show the effect of Coulomb deflection, since for the Al target they are steeper than those for the Ta target. If more nucleons are removed from the beam, the fragments, like $^{10}$Be, have wider momentum distributions peaked again near the beam velocity. Transfer reaction products have also been observed in the experiment. The momentum distributions of these fragments are wider than those of similar few nucleon removal products (e.g., $^{15}$C in contrast to $^{13}$C) and interestingly they are peaked at considerably lower velocities than that of the beam. This novel feature provides hints of a rather complex mechanism of

$$\frac{A}{q} = \frac{B \theta}{L} (T - \text{TOF}),$$

from which the $A/q$ parameter can be calculated also on an event-by-event basis.

![Figure 4](image-url)  
**FIG. 4.** Momentum distributions of some projectile-like isotopes produced from the reaction of $^{14}$N at 75 MeV/u with Al target at $0^\circ$ (closed points $\times 2$), $1^\circ$ (diamonds $\times 1$), and $2^\circ$ (squares $\pm 2$). The beam momentum is indicated by an arrow.

![Figure 5](image-url)  
**FIG. 5.** Angular distributions of some projectile-like isotopes produced from the reaction of $^{14}$N at 75 MeV/u with Al (squares) and Ta (closed points) targets. At each point, the angular bin has a full-width of $1^\circ$. The lines are just to guide the eye.
nucleon rearrangement between the projectile and the target and requires further systematic investigation.

ACKNOWLEDGMENTS

We wish to thank D. Mikolas and J. A. Winger for their help and suggestions during the course of the measurements.

APPENDIX: DERIVATION OF THE EXPRESSIONS FOR THE MOMENTUM AND THE REACTION ANGLE

The final position \( x \) of a particle at the front counter is related to the initial position \( x_0 \), initial angle \( \theta_0 \) and \( \delta \) with the expression [see Figs. 3(a) and 3(b)]:

\[
\delta = \frac{x - (x| x)x_0 - (x| \theta)\theta_0}{(x| \delta) + (x| x\delta)x_0 + (x| \theta\delta)\theta_0 + (x| \delta^2)\delta + (x| \theta\delta^2)\theta_0\delta}.
\]

Substituting \( \delta = 100[(\rho/\rho_0) - 1] \) in the left side of the above equation, omitting the \( (x| \delta^2) \) term (since it has a negligible contribution, see values in Table II), and using the abbreviated expressions:

\[
(x| \theta\delta)' = (x| \theta\delta) + (x| \theta\delta^2)\delta,
\]

\[
(x| \delta)' = (x| \delta) + (x| x\delta)x_0,
\]

we get

\[
\rho = \rho_0 + \frac{\rho_0}{100(x| \delta)'} \left[ (x| \theta\delta)'\theta_0 - (x| \theta)\theta_0 \right] - (x| \theta)\theta_0.
\]

Expanding the binomial in the denominator to first order in its argument (since it is small), we get:

\[
\rho = \rho_0 + \frac{\rho_0}{100(x| \delta)'} \left[ 1 + [(x| \theta\delta)'\theta_0/(x| \delta)'] \right] \left[ x - (x| x)x_0 \right] - (x| \theta)\theta_0.
\]

Now, referring to Fig. 3(b), the position coordinate \( x \) (with respect to the coordinate system \( xOz \) in the horizontal plane), is related to the measured position \( X_{FP} \) (in channels) as

\[
x = \frac{X_{FP} - X_{ref}}{C_{FP}} + \alpha,
\]

in which the negative sign is due to the fact that the direction of increasing position as measured by the detector was taken to be opposite to that indicated by the coordinate system \( xOz \). After substitution of this expression for \( x \) in the above equation and some algebraic manipulation, keeping terms up to first order in \( \theta_0 \), we arrive at the relation:

\[
\rho(\theta_0) = A_0(\theta_0) - A_1(\theta_0)X_{FP},
\]

where

\[
x = (x| x)x_0 + (x| \theta)\theta_0 + (x| \delta)\delta + (x| x\delta)x_0\delta
\]

\[
+ (x| \theta\delta)\theta_0\delta + (x| \delta^2)\delta^2 + (x| \theta\delta^2)\theta_0\delta^2,
\]

(A1)

where only the most important for the present system matrix elements are included (for example all \( \nu \)-dependent terms are dropped). After rearrangement, we can get the following equation for \( \delta \):

\[
A_0(\theta_0) = \left( \rho_0 + \frac{\rho_0}{100(x| \delta)'} X_{off} \right) - \frac{\rho_0}{100(x| \delta)'} \times \left( (x| \theta) + \frac{(x| \theta\delta)'}{(x| \delta)'} X_{off} \right) \theta_0
\]

and

\[
A_1(\theta_0) = \frac{\rho_0}{100(x| \delta)'C_{FP}} \left[ 1 - \frac{(x| \theta\delta)'}{(x| \delta)'} \theta_0 \right]
\]

with the definition:

\[
x_{off} = (X_{ref}/C_{FP}) + \alpha - (x| x)x_0.
\]

By writing the equations of \( A_0(\theta_0) \) and \( A_1(\theta_0) \) for \( \theta_0 = \theta_b \) and subtracting the latter from the former, we finally get:

\[
A_0(\theta_0) = A_0(\theta_b) - \frac{\rho_0}{100(x| \delta)'} \left( (x| \theta) + \frac{(x| \theta\delta)'}{(x| \delta)'} X_{off} \right) \times (\theta_0 - \theta_b)
\]

and

\[
A_1(\theta_0) = A_1(\theta_b) - \frac{\rho_0}{100C_{FP}} \left( x| \theta\delta \right)' \left( \theta_0 - \theta_b \right).
\]

In order to derive the equation for the scattering angle \( \theta_n \), we start from the expression that relates the final angle \( \theta \) of a particle at the front counter with the initial coordinates \( x_0, \theta_0, \) and \( \delta \):

\[
\theta = (\theta| x)x_0 + (\theta| \theta)\theta_0 + (\theta| \delta)\delta + (\theta| x\delta)x_0\delta
\]

\[
+ (\theta| \theta\delta)\theta_0\delta + (\theta| \delta^2)\delta^2 + (\theta| \theta\delta^2)\theta_0\delta^2.
\]

(A2)

Referring again to Fig. 3(b), we can write \( \theta = \theta_{FP} + \beta \), where \( \theta_{FP} \) is the angle of a given particle with respect to the angle of the reference 0° elastically scattered beam particles and \( \beta \) is the angle of these arbitrarily chosen ref-
ference beam particles with respect to the optical axis of the spectrometer. Substituting this into Eq. (A2) and rearranging we get

\[ \theta_{FP} = C_0(\theta_0) + C_1(\theta_0)\delta + C_2(\theta_0)\delta^2, \]

with:

\[ C_0(\theta_0) = -\beta + (\theta|\theta)x_0 + (\theta|\theta)\theta_0, \]
\[ C_1(\theta_0) = (\theta|\delta) + (\theta|x\delta)x_0 + (\theta|\theta\delta)\theta_0, \]
\[ C_2(\theta_0) = (\theta|\delta^2) + (\theta|\theta\delta^2)\theta_0. \]

By rewriting each of these equations for \( \theta_0 = \theta_b \) and subtracting from the corresponding equation for \( \theta_b \), we find:

\[ C_0(\theta_0) = C_0(\theta_b) + (\theta|\theta)(\theta_0 - \theta_b), \]
\[ C_1(\theta_0) = C_1(\theta_b) + (\theta|\theta\delta)(\theta_0 - \theta_b), \]
\[ C_2(\theta_0) = C_2(\theta_b) + (\theta|\theta\delta^2)(\theta_0 - \theta_b). \]

Substituting these expressions into the equation for \( \theta_{FP} \) and solving for \( \theta_b = \theta_0 - \theta_b \), we get:

\[ \theta_b = \frac{\theta_{FP} - [C_0(\theta_b) + C_1(\theta_b)\delta + C_2(\theta_b)\delta^2]}{(\theta|\theta) + (\theta|\theta\delta) + (\theta|\theta\delta^2)\theta_0}. \]

The aberration-limited momentum resolution of the system is given by:

\[ R_{ab} = \Delta x_{ab}/(x|\delta), \]  
(A3)

where \( \Delta x_{ab} \) is the image size, which can be estimated to a first approximation, by an addition in quadrature of the contributions from the different aberrations. For the present system, for a given value of \( \delta \), \( \Delta x_{ab} \) can be estimated as:

\[ \Delta x_{ab} = ([(x|\delta)\Delta x_0]^2 + [(x|\delta\delta)\Delta x_0\theta_0]^2 + [(x|\theta\delta)\theta_0\delta]^2)^{1/2}. \]  
(A4)

The scattering angle uncertainty \( d\theta_0 \) can be estimated, to a first approximation, as:

\[ d\theta_0 = \frac{1}{(\theta|\theta)} \{d\theta_{FP}^2 + [(\theta|x)\Delta x_0]^2\}^{1/2}, \]  
(A5)

where \( d\theta_{FP} \) is the uncertainty in the measured final angle \( \theta_{FP} \).

Expressions (A4) and (A5) follow from Eqs. (A1) and (A2), respectively, where the most important terms are kept and a quadratic addition of the corresponding errors is performed.


